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OSCILLATION AND NONOSCILLATION OF NONLINEAR SECOND ORDER DIFFERENCE EQUATIONS

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ABSTRACT. In this paper, we study oscillation and nonoscillation behaviour of the second order nonlinear difference equations of the form

$$\Delta(r_n\psi(x_n)\Delta x_n) + q_{n+1}f(x_{n+1}) = 0, \qquad n \in N(n_o),$$

and

$$\Delta(r_n\psi(x_n)\Delta x_n) + q_n\, f(n, x_n) = 0, \qquad n \in N(n_o),$$

where $N(n_o) = \{n_o, n_o + 1, ...\}$, ($n_o$ is a fixed nonnegative integer number), $\Delta x_n = x_{n+1} - x_n$ is the forward difference operator, $x : N(n_o) \to \mathbb{R}$, $r : N(n_o) \to (0, \infty)$, $\psi : \mathbb{R} \to (0, \infty)$, f is a real valued continuous function, and $\{q_n\}$ is a sequence of real valued.

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1. Introduction

In recent years, there has been an increasing interest in the study of oscillation and asymptotic behaviour of solutions of nonlinear difference equations, see for example ([1], [3], [4], [10], [11], [13], [14]) and the references cited therein. In [3], [7] and [8], the authors have dealt with oscillation of the difference equation

$$\Delta(r_n\Delta x_n) + f(n, x_n) = 0, \quad n \in N(n_o).$$

The aim of this paper is to obtain a new criteria for oscillation and nonoscillation of the general difference equations

$$\Delta(r_n\psi(x_n)\Delta x_n) + q_{n+1}f(x_{n+1}) = 0, \qquad n \in N(n_o), \tag{1}$$

and

$$\Delta(r_n\psi(x_n)\Delta x_n) + q_nf(n, x_n) = 0, \qquad n \in N(n_o), \tag{2}$$

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where $N(n_o) = \{n_o, n_o+1, ...\}$, ($n_o$ is a fixed nonnegative integer number), $\Delta x_n = x_{n+1} - x_n$ is the forward difference operator, $x : N(n_o) \to \mathbb{R}$, $r : N(n_o) \to (0,\infty)$, $\psi : \mathbb{R} \to (0,\infty)$, f is a real valued continuous function and $\{q_n\}$ is a sequence of real valued. Section 1 consists of a brief introduction and review of relevant material. In Section 2 we discuss a new sufficient condition for oscillation of all solutions of the second order nonlinear difference equations of type (1). In section 3 we present several necessary and sufficient conditions for nonoscillation of solutions of (2). A nontrivial solution of (1) or (2) is said to be oscillatory if for every $n_o \in N(n_o)$ there exists $n \geq n_o$ such that $x_n\, x_{n+1} < 0$ ([1], p. 322). Otherwise, it is called nonoscillatory.

2. Oscillation of nonlinear difference equations

In this section we give sufficient conditions for oscillation of solutions of equation (1) with oscillating coefficients q_n. Our results in this section improve and partially generalize some results of Thandapani, et al [9] and Zhang, et al [14].

Through this section, we assume that

(I) $\quad f : \mathbb{R} \to \mathbb{R}$ is nondecreasing function, $\quad xf(x) > 0, \quad x \neq 0.$

(II) $$\lim_{n\to\infty} \sum_{l=n_o}^{n} \frac{1}{r_l \psi(x_l)} = \infty, \quad \text{for} \quad n \in N(n_o).$$

The following Lemmas will be needed in this section.

Lemma 1. *Suppose that $\{x_n\}$, $n \in N(n_o)$, is a nonoscillatory solution of (1). If there exists an $n_1 \in N(n_o)$ such that*

$$-\frac{r_{n_o}\psi(x_{n_o})\Delta x_{n_o}}{f(x_{n_o})} + \sum_{l=n_o}^{n-1} q_l + \sum_{l=n_o}^{n_1-1} \frac{r_l\psi(x_l)\Delta x_l \Delta f(x_l)}{f(x_l)f(x_{l+1})} \geq m, \tag{3}$$

where $m > 0$ and $n \in N(n_o)$. Then

(1) $r_n\psi(x_n)\Delta x_n \leq -m\, f(x_{n_1})$, *when $\{x_n\}$ is a positive, $n \in N(n_1)$,* (4)

(2) $r_n\psi(x_n)\Delta x_n \geq -m\, f(x_{n_1})$, *when $\{x_n\}$ is a negative, $n \in N(n_1)$.* (5)

Proof. From (1), it is clear that

$$\frac{\Delta(r_n\psi(x_n)\Delta x_n)}{f(x_{n+1})} = -\,q_{n+1}, \quad \text{for} \quad n \in N(n_o).$$

Then

$$\Delta\left[\frac{r_n\psi(x_n)\Delta x_n}{f(x_n)}\right] = -\,q_{n+1} - \frac{r_n\psi(x_n)\Delta x_n \Delta f(x_n)}{f(x_n)f(x_{n+1})}. \tag{6}$$

Summing (6) from n_o to $n-1$, we have

$$\sum_{l=n_o}^{n-1} \Delta\left[\frac{r_l\psi(x_l)\Delta x_l}{f(x_l)}\right] = -\sum_{l=n_o}^{n-1} q_{l+1} - \sum_{l=n_o}^{n-1} \frac{r_l\psi(x_l)\Delta x_l \Delta f(x_l)}{f(x_l)f(x_{l+1})},$$

we get

$$-\frac{r_n\psi(x_n)\Delta x_n}{f(x_n)} = -\frac{r_{n_o}\psi(x_{n_o})\Delta x_{n_o}}{f(x_{n_o})} + \sum_{l=n_o}^{n-1} q_{l+1} + \sum_{l=n_o}^{n-1} \frac{r_l\psi(x_l)\Delta x_l \Delta f(x_l)}{f(x_l)f(x_{l+1})}. \tag{7}$$

By (3) and (7), we get

$$-\frac{r_n\psi(x_n)\Delta x_n}{f(x_n)} \geq m + \sum_{l=n_1}^{n-1} \frac{r_l\psi(x_l)\Delta x_l \Delta f(x_l)}{f(x_l)f(x_{l+1})}. \tag{8}$$

But since by (I), it follows that the sum of the right hand side of (8) is positive. Hence

$$x_n\Delta x_n < 0, \quad \text{for} \quad n \in N(n_1).$$

Now we have one of the two possibilities $\{x_n\}$ is positive or negative. Suppose first that $\{x_n\}$ is positive. Setting $-r_n\psi(x_n)\Delta x_n = w_n > 0$. Hence (8) becomes

$$\frac{w_n}{f(x_n)} \geq m - \sum_{l=n_1}^{n-1} \frac{w_l \Delta f(x_l)}{f(x_l)f(x_{l+1})}. \tag{9}$$

Now suppose that

$$\frac{v_n}{f(x_n)} = m - \sum_{l=n_1}^{n-1} \frac{v_l \Delta f(x_l)}{f(x_l)f(x_{l+1})}. \tag{10}$$

Furthermore, using induction, we can prove that $w_n \geq v_n$ for all $n \in N(n_1)$. Taking the difference operator on both sides of (10), we find

$$\Delta\left(\frac{v_n}{f(x_n)}\right) = \frac{\Delta v_n}{f(x_{n+1})} + v_n\Delta\left(\frac{1}{f(x_n)}\right) = -\frac{v_n\Delta f(x_n)}{f(x_n)f(x_{n+1})}.$$

Hence

$$\frac{\Delta v_n}{f(x_{n+1})} = 0, \quad f(x_{n+1}) \neq 0.$$

This implies that $\Delta v_n = 0$. Therefore $v_n = v_{n_1} = m\, f(x_{n_1})$, for $n \in N(n_1)$. Hence

$$r_n\psi(x_n)\Delta x_n \leq -m\, f(x_{n_1}), \quad n \in N(n_1).$$

The proof for the case when $\{x_n\}$ is negative follows from similar arguments by taking $r_n\psi(x_n)\Delta x_n = w_n > 0$. □

Theorem 1. *Let $\{x_n\}$, $n \in N(n_1-1)$, be a solution of Eq (1)*

(i) *If* $\{x_n\}$ *is a nonoscillatory solution of Eq (1) and* $\lim_{n\to\infty}\inf \sum_{l=n_1}^{n} q_l > -\infty$, *then*

$$\sum_{l=n_1}^{\infty} \frac{r_l\psi(x_l)\Delta x_l \Delta f(x_l)}{f(x_l)f(x_{l+1})} < \infty. \tag{11}$$

(ii) *If* $\lim_{n\to\infty} \sum_{l=n_1}^{n} q_l = \infty$, *then every solution of Eq (1) is oscillatory.*

Proof. (i) For the sake of contradiction, assume that

$$\sum_{l=n_1}^{\infty} \frac{r_l\psi(x_l)\Delta x_l \Delta f(x_l)}{f(x_l)f(x_{l+1})} = \infty.$$

Since by assumption $\lim_{n\to\infty}\inf \sum_{l=n_1}^{n} q_l > -\infty$, then there exists $n_1^* \geq n_1$ such that (3) holds. For the case $\{x_n\}$ is positive and by Lemma 1, we obtain

$$r_n\psi(x_n)\Delta x_n \leq -m\, f(x^*_{n_1}), \quad \text{for} \quad n \geq n_1^*.$$

But since $m > 0$ and $\psi(x_n)$ is positive for $n \in N(n_1^*)$, then we have

$$\Delta x_n \leq -m\, f(x_{n_1^*}) \left(\frac{1}{r_n\psi(x_n)} \right).$$

Then

$$\sum_{l=n_1^*}^{n-1} \Delta x_n \leq -m\, f(x_{n_1^*}) \sum_{l=n_1^*}^{n-1} \frac{1}{r_l\psi(x_l)},$$

i.e.,

$$x_n \leq x_{n_1^*} - m\, f(x_{n_1^*}) \sum_{l=n_1^*}^{n-1} \frac{1}{r_l\psi(x_l)}. \tag{12}$$

The right hand side of (12) tends to $-\infty$ as $n \to \infty$, while the left side is positive. This is a contradiction. The proof for the case $\{x_n\}$ is negative is similar.

(ii) Suppose the contrary that, there exists a positive nonoscillatory solution of (1) say $\{x_n\}$ for all $n \geq n_1$, then the condition of Lemma 1 is satisfied. Thus we have

$$\Delta x_n \leq -\frac{m\, f(x_{n_1})}{r_n\psi(x_n)}, \quad \text{for} \quad n \geq n_1. \tag{13}$$

Now taking the sum of (13), from n_1 to $n-1$, we get

$$x_n \leq x_{n_1} - m\, f(x_{n_1}) \sum_{l=n_1}^{n-1} \frac{1}{r_l\psi(x_l)}. \tag{14}$$

Taking the limit of (14) as $n \to \infty$, we get $x_n \to -\infty$. This is a contradiction. The case of $x_n < 0$, the proof is similar and hence it is omitted. □

Lemma 2. *Assume that*

$$(A_1)\ \lim_{|x|\to\infty} |f(x)| = \infty, \qquad (A_2)\ \lim_{n\to\infty} \sum_{l=n_o}^{n} q_l \ \text{exists.}$$

If $\{x_n\}$ is a nonoscillatory solution of (1). Then

$$\frac{r_n \psi(x_n)\Delta x_n}{f(x_n)} = \sum_{l=n}^{\infty} q_{l+1} + \sum_{l=n}^{\infty} \frac{r_l \psi(x_l)\Delta x_l \Delta f(x_l)}{f(x_l) f(x_{l+1})}, \ \text{for } n \in N(n_o). \quad (15)$$

Proof. The proof is similar to the proof of Lemma 2.2 in [14], and so it is omitted. □

Theorem 2. *Let (A_2) be satisfied. Suppose that*

$$(A_3)\ 0 < \int_{\epsilon}^{\infty} \frac{dy}{f(y)}, \quad \text{and} \quad \int_{-\epsilon}^{-\infty} \frac{dy}{f(y)} < \infty, \ \text{for all } \epsilon > 0;$$

$$(A_4)\ \lim_{n\to\infty} \sum_{l=n_o}^{n} \frac{1}{r_l \psi(x_l)} \sum_{i=l+1}^{\infty} q_i = \infty.$$

Then every solution of (1) is oscillatory.

Proof. Suppose this is false. Without loss of generality, let $\{x_n\}$ be an eventually positive solution of (1). Then by Lemma 2 and condition (I), we obtain

$$\sum_{l=n}^{\infty} \frac{r_l \psi(x_l)\Delta x_l \Delta f(x_l)}{f(x_l) f(x_{l+1})} \geq 0.$$

Thus

$$\frac{r_n \psi(x_n)\Delta x_n}{f(x_n)} \geq \sum_{l=n}^{\infty} q_{l+1}, \quad \text{for} \quad n \in N(n_o),$$

i.e.,

$$\frac{\Delta x_n}{f(x_n)} \geq \frac{1}{r_n \psi(x_n)} \sum_{l=n}^{\infty} q_{l+1}, \quad \text{for} \quad n \in N(n_o). \quad (16)$$

The sum of both sides of (16) from n_o to n, we obtain

$$\sum_{l=n_o}^{n} \frac{\Delta x_l}{f(x_l)} \geq \sum_{l=n_o}^{n} \frac{1}{r_l \psi(x_l)} \sum_{i=l+1}^{\infty} q_{i+1}. \quad (17)$$

Define $g(t) = x_l + (t-l)\Delta x_l,\ l \leq t \leq l+1$. Then we have one of the following two cases.

Case 1. If $\Delta x_l \geq 0$, then $x_l \leq g(t) \leq x_{l+1}$. Thus in view of the assumption (I), we get

$$\frac{\Delta x_l}{f(x_{l+1})} \leq \frac{g'(t)}{f(g(t))} \leq \frac{\Delta x_l}{f(x_l)}. \tag{18}$$

Case 2. If $\Delta x_l < 0$, then $x_{l+1} \leq g(t) \leq x_l$. So we can directly obtain (18).

Now by (17) and (18), we get

$$\int_{g(n_o)}^{\infty} \frac{ds}{f(s)} \geq \int_{n_o}^{n+1} \frac{dg(t)}{f(g(t))} \geq \sum_{l=n_o}^{n} \frac{1}{r_l \psi(x_l)} \sum_{i=l+1}^{\infty} q_{i+1}. \tag{19}$$

Let $G(y) = \displaystyle\int_y^{\infty} \frac{dy}{f(y)}$, then

$$G(g(n_o)) \geq \sum_{l=n_o}^{n} \frac{1}{r_l \psi(x_l)} \sum_{i=l+1}^{\infty} q_{i+1}.$$

This contradicts condition (A_4). Similarly, one can prove that (3) does not possess eventually negative solution. □

3. Nonoscillatory behaviour of solutions

In this section, we discuss nonoscillatory behaviour of solutions of (2). We assume that $\psi(x)$ is nondecreasing in x. Let $\{q_n\}_{n=n_o}^{\infty}$ be a positive sequence of real numbers. Our results partially generalize those of [7].

Through this section, we assume that the condition (II) holds, and

$$\text{(III)} f(n,x) > 0 \ \text{ for all } \ (n,x) \in N(n_o) \times (0, \infty).$$

Before stating our results we give the following result of [2] which considered as a discrete analog of Schauder's theorem.

Lemma 3. ([2]) *Let k be a closed and convex subset of l^∞. Suppose that T is a continuous map such that $T(k)$ is contained in k, and suppose further that $T(k)$ is uniformly Cauchy. Then T has a fixed point in k.*

Now we give the following results.

Lemma 4. *Let $\{x_n\}_{n=n_o}^{\infty}$ be an eventually positive solution of (2). Then there exist two positive constants c_1, c_2 and $s \in N(n_o)$ such that $\{x_n\}$ is monotonically increasing and*

$$c_1 \leq x_n \leq c_2 \mathbb{R}_{s,n}(c_1), \ \textit{for} \ \ n \in N(s),$$

where

$$\mathbb{R}_{s,n}(c_1) = \sum_{k=s}^{n-1} \frac{1}{r_k \psi(c_1)}.$$

Proof. Since $\{x_n\}$ is an eventually positive solutions of (2). Then there exists an $s \in N(n_o)$ such that $x_n > 0$ for $n \in N(s)$. It follows from (2) and (III) that

$$\Delta(r_n\psi(x_n)\Delta x_n) < 0,$$

i.e.,

$$x_{n+2} < x_{n+1} + (\Delta x_n)\frac{r_n\psi(x_n)}{r_{n+1}\psi(x_{n+1})}, \quad n \in N(s), \tag{20}$$

If there exists an $n_1 \in N(s)$ such that $x_{n_1+1} \leq x_{n_1}$, then it follows from (20) that $x_{n+2} < x_{n+1}$ for all $n \in N(n_1)$. This means that $\{x_n\}$ is eventually decreasing. But since by (20) and (II), it follows that

$$x_n \leq x_s + r_{s-1}\psi(x_{s-1})\Delta x_{s-1}\sum_{k=s}^{n-1}\frac{1}{r_k\psi(x_k)}. \tag{21}$$

It follows by (II) that $\{x_n\}$ is an eventually negative when n is large enough. This contradicts the fact that $\{x_n\}$ is eventually positive. Therefore $\{x_n\}$ is eventually increasing and the sequence $\{r_n\psi(x_n)\Delta x_n\}$ is positive. Since $x_n > 0$, and $\psi(x_n)$ is nondecreasing for $n \in N(s)$, then there exists $c_1 > 0$ such that $c_1 \leq x_n$ for $n \in N(s)$. Thus

$$\sum_{k=s}^{n-1}\frac{1}{r_k\psi(x_k)} \leq \sum_{k=s}^{n-1}\frac{1}{r_k\psi(c_1)}. \tag{22}$$

Therefore, the relation (21) becomes

$$x_n \leq x_s + r_{s-1}\psi(x_{s-1})\Delta x_{s-1}\sum_{k=s}^{n-1}\frac{1}{r_k\psi(c_1)}, \tag{23}$$

i.e.,

$$x_n \leq c_2\mathbb{R}_{s,\,n}(c_1),$$

where c_2 is a positive constant. □

Theorem 3. *If any nonoscillatory positive solution of (2) belongs to K_α^0, and $f(n,x)$ is nonincreasing, then*

$$\sum_{k=n_o}^{\infty}\frac{1}{r_k\psi(c)}\sum_{i=k}^{\infty}q_i f(i,c) < \infty, \quad \textit{for} \quad c > 0. \tag{24}$$

where $K_\alpha^0:\ x_n \to \alpha, \qquad r_n\psi(x_n)\Delta x_n \to 0, \qquad (n \to \infty).$

Proof. Let $\{x_n\}$ be any nonoscillatory positive solution of (2) belongs to K_α^0. Since $x_n > 0$, then $\alpha > 0$, and there exist two positive constants $c_1,\ c_2$ and an $s \in N(n_o)$ such that

$$c_1 \leq x_n \leq c_2 \quad \text{for all} \quad n \in N(s). \tag{25}$$

On the other hand, by (2) we get,

$$r_n\psi(x_n)\Delta x_n + \sum_{k=m}^{n-1} q_k f(k, x_k) = r_m\psi(x_m)\Delta x_m, \tag{26}$$

for $m \in N(s)$ and $n \in N(m)$. Taking the limit as $n \to \infty$ on both sides of (26), and using K_α^0, we obtain

$$r_m\psi(x_m)\Delta x_m = \sum_{k=m}^{\infty} q_k f(k, x_k). \tag{27}$$

But since by (27) we get,

$$x_m = x_s + \sum_{k=s}^{m-1} \frac{1}{r_k\psi(x_k)} \sum_{i=k}^{\infty} q_i f(i, x_i). \tag{28}$$

Thus it follows by taking the limit as $m \to \infty$ on both sides of (28) that

$$\sum_{k=s}^{\infty} \frac{1}{r_k\psi(x_k)} \sum_{i=k}^{\infty} q_i f(i, x_i) < \infty. \tag{29}$$

But since $x_n > 0$, f is nonincreasing, and ψ is nondecreasing. Thus by (25) we have

$$\sum_{i=k}^{\infty} q_i f(i, c_2) \le \sum_{i=k}^{\infty} q_i f(i, x_i), \quad \text{and} \quad \frac{1}{r_k\psi(c_2)} \le \frac{1}{r_k\psi(x_k)}$$

Then

$$\sum_{k=n_o}^{\infty} \frac{1}{r_k\psi(c_2)} \sum_{i=k}^{\infty} q_i f(i, c_2) \le \sum_{k=n_o}^{\infty} \frac{1}{r_k\psi(x_k)} \sum_{i=k}^{\infty} q_i f(i, x_i),$$

i.e., the inequality (24) holds. □

Theorem 4. *Assume that $f(n, x)$ is nonincreasing. If*

$$\sum_{k=n_o}^{\infty} q_k f(k, a) \ < \ \infty, \ \textit{for some} \ a > 0, \tag{30}$$

and

$$\sum_{k=n_o}^{\infty} \frac{1}{r_k\psi(b)} \sum_{i=k}^{\infty} q_i f(i, b) \ = \ \infty, \quad \textit{for some} \ b > 0. \tag{31}$$

Then Eq (2) has a nonoscillatory positive solution of class K_∞^0, where
$K_\infty^0: \ x_n \to \infty, \qquad r_n\psi(x_n)\Delta x_n \to 0, \qquad (n \to \infty).$

Proof. Introduce the linear space X of all real sequence $\{x_n\}$ such that

$$\sup_{n\in N(n_o)} \frac{|x_n|}{R_{n_o,n}(a)} < \infty, \quad \text{where } R_{n_o,n}(a) = \sum_{k=n_o}^{n-1} \frac{1}{r_k \psi(a)}.$$

It is not difficult to see that

$$\|x\| = \sup_{n\in N(n_o)} \frac{|x_n|}{R_{n_o,n}(a)}, \qquad x \in X, \tag{32}$$

is a Banach space [6, 8, 12]. Consider the subset φ of X consisting of all $x \in X$, such that

$$\varphi = \left\{ x \in X \mid a \le x_n \le a + \sum_{k=n_o}^{n-1} \frac{1}{r_k \psi(a)} \sum_{i=k}^{\infty} q_i f(i,a),\ n \ge n_o \right\}.$$

We also define an operator $T : \varphi \to \varphi$ by the formula

$$(Tx)_n = a + \sum_{k=n_o}^{n-1} \frac{1}{r_k \psi(x_k)} \sum_{i=k}^{\infty} q_i f(i, x_i). \tag{33}$$

The mapping T satisfies the assumption of the Shauder's fixed point theorem. Namely, it satisfies the following

(1) T maps φ into φ.
(2) T is continuous.

In fact , if for $\epsilon > 0$, we choose $s \ge n_o$ so large such that

$$\sum_{k=s}^{\infty} q_k\, f(k,a) < \frac{\epsilon}{2}, \quad \text{for all} \quad n \in N(s). \tag{34}$$

Let $\{x_n^v\}_{v=1}^{\infty}$ be a sequence of elements of φ such that $x^v \to x$ as $v \to \infty$. Hence since φ is closed, $x \in \varphi$, for all large v it follows that

$$\left| \frac{(Tx^v)_n}{R_{n_o,n}(a)} - \frac{(Tx)_n}{R_{n_o,n}(a)} \right| \le \left| \frac{1}{R_{n_o,n}(a)} \sum_{k=n_o}^{n-1} \frac{1}{r_k \psi(x_k^v)} \sum_{i=k}^{\infty} q_i f(i, x_i^v) \right| + \left| \frac{1}{R_{n_o,n}(a)} \sum_{k=n_o}^{n-1} \frac{1}{r_k \psi(x_k)} \sum_{i=k}^{\infty} q_i f(i, x_i) \right|$$

Since $x \in \varphi$, $f(n,x)$ is nonincreasing, $\psi(x)$ is nondecreasing positive value and $a \le x_n$, then $\psi(a) \le \psi(x_n)$ implies that $\frac{1}{\psi(a)} \ge \frac{1}{\psi(x_n)}$ and $f(n,x_n) \le f(n,a)$. We have

$$\left| \frac{(Tx^v)_n}{R_{n_o,n}(a)} - \frac{(Tx)_n}{R_{n_o,n}(a)} \right| \le \sum_{k=n}^{\infty} q_k\, f(k, x_k^v) + \sum_{k=n}^{\infty} q_k\, f(k, x_k) \le 2 \sum_{k=n}^{\infty} q_k\, f(k,a).$$

This shows that $\lim_{v\to\infty}\|Fx^v - Fx\| = 0$; i.e., T is continuous.

(3) $T\varphi$ is uniformly Cauchy.

Let $x \in \varphi$ and $m,\ n \geq n_o$

$$\begin{aligned}\left|\frac{(Tx)_m}{R_{n_o,m}(a)} - \frac{(Tx)_n}{R_{n_o,n}(a)}\right| &\leq \left|\frac{a}{R_{n_o,m}(a)} - \frac{a}{R_{n_o,n}(a)}\right| \\ &\quad + \left|\frac{1}{R_{n_o,m}(a)}\sum_{k=n_o}^{m-1}\frac{1}{r_k\psi(x_k)}\sum_{i=k}^{\infty} q_i f(i,x_i)\right. \\ &\quad \left. - \frac{1}{R_{n_o,n}(a)}\sum_{k=n_o}^{n-1}\frac{1}{r_k\psi(x_k)}\sum_{i=k}^{\infty} q_i f(i,x_i)\right| \\ &\leq \frac{2a}{R_{n_o,n}(a)} + \frac{1}{R_{n_o,m}(a)}\sum_{k=n_o}^{m-1}\frac{1}{r_k\psi(x_k)}\sum_{i=k}^{\infty} q_i f(i,x_i) \\ &\quad + \frac{1}{R_{n_o,n}(a)}\sum_{k=n_o}^{n-1}\frac{1}{r_k\psi(x_k)}\sum_{i=k}^{\infty} q_i f(i,x_i) \\ &\leq \frac{2a}{R_{n_o,n}(a)} + 2\sum_{k=n}^{\infty} q_k\, f(k,a).\end{aligned}$$

Since $R_{n_o,n}(a) \to \infty$ as $n \to \infty$.

Hence, for any $\epsilon > 0$, there exists an integer $n_1 \in N(n_o)$ such that, for $n \in N(n_1)$,

$$\frac{a}{R_{n_o,n}(a)} < \frac{\epsilon}{4}, \quad \text{and} \quad \sum_{k=n_1}^{\infty} q_k\, f(k,a) < \frac{\epsilon}{4}.$$

Thus,

$$\left|\frac{(Tx)_m}{R_{n_o,m}(a)} - \frac{(Tx)_n}{R_{n_o,n}(a)}\right| < \epsilon,$$

for $m > n \geq n_1$. This means that $T\varphi$ is uniformly Cauchy.

By Lemma 3, we can conclude that there exists, an $x \in \varphi$ such that $x = Tx$; that is, $\{x_n\}$ is a positive solution of (2). Taking the difference operator on both sides of (33), we get

$$r_n\psi(x_n)\Delta x_n = \sum_{k=n}^{\infty} q_k f(k,x_k).$$

Hence, $\lim\limits_{n\to\infty} r_n\psi(x_n)\Delta x_n = 0$, which implies that $\{x_n\}$ is increasing for $n \geq n_o$, and $\{x_n\}$ either converges to some positive limit or diverges to ∞ as $n \to \infty$. Suppose that the first case hold. Then, this means that $x \in K_\alpha^0$, and so (24) holds. But this contradicts the assumption (31). Then $\{x_n\}$ is a positive solution of (2) belongs to K_∞^0. □

Theorem 5. *If Eq (2) has a nonoscillatory positive solution of class K^0_∞, and $f(n,x)$ is nonincreasing, then*

$$\sum_{k=n_o}^{\infty} q_k f(k, aR_{n_o,k}(b)) < \infty, \tag{35}$$

and

$$\sum_{k=n_o}^{\infty} \frac{1}{r_k\psi(b)} \sum_{i=n}^{\infty} q_i f(i,b) = \infty, \tag{36}$$

where $a,\ b > 0,\ n \geq n_o$, *and* $R_{n_o,n}(b) = \sum_{k=n_o}^{n-1} \frac{1}{r_k\psi(b)}$.

Proof. Since $x_n > 0$ and $x \in K^0_\infty$. For any $a > 0$ and $b > 0$, by Lemma 4, there exists $m \in N(n_o)$ such that $b \leq x_n \leq aR_{n_o,n}(b)$, for all $n \in N(m)$. Since, f is nonincreasing, then

$$f(n,x_n) \leq f(n,b), \quad \text{and} \quad f(n,x_n) \geq f(n, aR_{n_o,n}(b)),$$

for $n \in N(m)$. Thus by Eq (2), we obtain

$$r_n\psi(x_n)\Delta x_n + \sum_{k=m}^{n-1} q_k f(k,x_k) = r_m\psi(x_m)\Delta x_m, \quad \text{for} \quad n \in N(m). \tag{37}$$

Taking the limit of (37) as $n \to \infty$. Hence since $x \in K^0_\infty$, we have

$$r_m\psi(x_m)\Delta x_m = \sum_{k=m}^{\infty} q_k f(k,x_k), \tag{38}$$

since $x_n \leq aR_{n_o,n}(b)$ and $f(n,x_n) \geq f(n, aR_{n_o,n}(b))$, then

$$r_m\psi(x_m)\Delta x_m \geq \sum_{k=m}^{\infty} q_k f(k, aR_{n_o,n}(b))$$

This means that (35) holds. Now since by (38), we have

$$x_m = x_{n_o} + \sum_{k=n_o}^{m_1} \frac{1}{r_k\psi(x_k)} \sum_{i=k}^{\infty} q_i f(i,x_i) \quad \text{for} \quad m \in N(n_o). \tag{39}$$

Taking the limit as $m \to \infty$, on both sides of (39), it follows that

$$\sum_{k=n_o}^{\infty} \frac{1}{r_k\psi(x_k)} \sum_{i=k}^{\infty} q_i f(i,x_i) = \infty.$$

But since $f(n,x_n) \leq f(n,b)$, and $\frac{1}{r_k\psi(x_k)} \leq \frac{1}{r_k\psi(b)}$, then we get (36). □

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